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Large N Solution of the 2D Supersymmetric Yang-Mills Theory

Miao Li
Department of Physics
Brown University
Providence, RI 02912
li@het.brown.edu

The Schwinger-Dyson equations of the Makeenko-Migdal type, when supplemented with some simple equations as consequence of supersymmetry, form a closed set of equations for Wilson loops and related quantities in the two dimensional super-gauge theory. We solve these equations. It appears that the planar Wilson loops are described by the Nambu string without folds. We also discuss how to put the model on a spatial lattice, where a peculiar gauge is chosen in order to keep one supersymmetry on the lattice. Supersymmetry is unbroken in this theory. We comment on possible generalization of these considerations to other models.

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1. Introduction

The large N problem of usual gauge theory remains a formidable problem, despite the existence of a closed set of large N equations, the well-known Makeenko-Migdal equations [1]. Some progress has been made recently by Migdal [3], although it seems that additional input is needed in order to finally solve these equations. Interestingly enough, these equations can be solved in two dimensions, as shown long ago by Kazakov and Kostov [2]. Unfortunately, as soon as one introduces dynamical scalar particles or quarks in the adjoint representation, the large N problem again becomes intractable. For some references on this topic, see [4]. In light of recent progress in four dimensional supersymmetric Yang-Mills theories, initiated in the work of Seiberg-Witten [5] and furthered in [6] and [7], it is tempting to ask whether supersymmetry helps in solving the large N problem. Our goal in this paper is a modest one, instead of working in four dimensions, we ask whether supersymmetry helps in solving the large N problem in two dimensions, where introducing dynamical adjoint matter already complicates the problem a lot. The answer is yes, though the method adopted here is completely different from that of [5]. The fact that our model is the dimensional reduction of a three dimensional $N = 1$ super-gauge theory or even of a four dimensional super-gauge theory may hint at possible simplification of the large N problem in these models.

If one starts with deriving an equation in the super-gauge model parallel to the ordinary Makeenko-Migdal equation, one need to derive more equations in order to get a closed system. One soon realizes that infinitely many equations are needed, so this way of proceeding is hopeless. As it turns out, the only equations we need to supplement the MM equations are the ones resulting from the Ward identities associated with supersymmetry. These identities are valid only when supersymmetry is not dynamically broken. This will be demonstrated in sect.4, where we put the model in a spatial box.

In the pure gauge theory, as being solved in [2], the only relevant modes are topological. The solution of Wilson loops with intersections is quite nontrivial. In addition to the usual area law, the dependence of these loops on areas of windows is polynomial without a definite sign. This implies that if one tries to formulate any string theory (as attempted at in [8]), one would have to introduce fermions on the world sheet. Indeed a formulation of such theory has proven quite unwieldy. It may appear surprising that the solution of Wilson loops in the supersymmetric theory is simpler than that in the pure gauge theory, as will be seen in sect.3. It appears that the planar Wilson loops are described by the Nambu string without folds. It remains to see whether at the string loop level (1/N corrections) the correspondence persists. In any case, our result already indicates the following interesting picture. In two dimensions, when there is only gauge field, namely the theory is purely

bosonic, then one has to introduce fermionic degrees on the world sheet. However in the super-gauge theory, where there is a fermion in spacetime, one has only bosonic degrees of freedom on the world sheet (the fold-less constraint can be easily implemented). Further study is necessary to understand other loop-like quantities in addition to the Wilson loop.

To begin with, let us write down the action of the $N=1$ supersymmetric Yang-Mills theory in which the super-multiplet consists of a gauge field A_μ, an adjoint scalar ϕ, and an adjoint real fermion λ, each field is a Hermitian matrix. Here for simplicity we consider a $U(N)$ gauge group which makes no difference than a gauge group $SU(N)$ in the large N limit. We shall follow conventions in [9]. Let $\sigma^0=\bar\sigma^0=-1$, $\sigma^1=-\bar\sigma^1$ one of the Pauli matrices. The supersymmetric action is

$$S=\int d^2x \mathrm{tr}\,\left(-\frac{1}{4}F^2_{\mu\nu}-\frac{1}{2}(D_\mu\phi)^2-i\lambda\bar\sigma^\mu D_\mu\lambda-g\lambda\sigma^3[\phi,\lambda]\right), \tag{1.1}$$

where g is the coupling constant. The field strength is defined according to $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu]$ and the covariant derivative of a Hermitian matrix field A (in (1.1) it is either ϕ or λ) is defined by $D_\mu A=\partial_\mu A+ig[A_\mu,A]$. The action (1.1) is invariant under the following supersymmetry transformation

$$\begin{aligned}\delta A_\mu&=-2i\lambda\bar\sigma_\mu\epsilon,\\ \delta\phi&=2i\lambda\sigma^3\epsilon,\\ \delta\lambda&=\sigma^1F_{01}\epsilon-\sigma^\mu\sigma^3D_\mu\phi\epsilon.\end{aligned} \tag{1.2}$$

For large N considerations, it is often convenient to rescale all fields such that the action is weighted by a factor N, also we need to hold g^2N fixed for large N. Thus let $g\sqrt{N}\to g$, and $A_\mu\to(1/g)A_\mu\to(\sqrt{N}/g)A_\mu$, $\lambda\to(\sqrt{N}/g)\lambda$ and $\phi\to(\sqrt{N}/g)\phi$. Since all fields are rescaled by the same factor, the transformation law in (1.2) remains the same, while the action is now weighted by a overall factor N/g^2:

$$S=\frac{N}{g^2}\int d^2x \mathrm{tr}\,\left(-\frac{1}{4}F^2_{\mu\nu}-\frac{1}{2}(D_\mu\phi)^2-i\lambda\bar\sigma^\mu D_\mu\lambda-\lambda\sigma^3[\phi,\lambda]\right), \tag{1.3}$$

In the definition of the field strength and the covariant derivatives there is no explicit dependence on g. Now it is g^2 not g^2N held fixed in the limit $N\to\infty$.

We will work in Minkowski spacetime throughout this paper.

The plan for the rest of the paper is as follows. In sect.2 we shall consider a set of Makeenko-Migdal equations, and a Ward identity associated to supersymmetry. This Ward identity, together with one of Makeenko-Migdal equations, does not yet form a closed set of equations for the Wilson loop and a quantity with two insertions of the fermion

field and a third quantity. The validity of the Ward identity depends upon unbroken supersymmetry, which we will prove in sect.4. We then proceed in sect.3 to argue that the third quantity is indeed vanishing, so we have a closed set of equations. These equations are easily solved. Sect.4 can be ignored if the reader does not wish to read the proof of unbroken SUSY. A special gauge is chosen in sect.4 to discuss the Hamiltonian formulation of the theory, in order to keep a supersymmetry on a spatial lattice. For gauge group $SU(N)$, the Witten index is calculated to be tr $(-1)^F = 1$ or tr $(-1)^F = (-1)^N$. The ambiguity in determining the sign of the index is discussed and resolved. In either case, it is nonvanishing and signaling unbroken SUSY. Sect.5 is devoted to a discussion. The model studied in this paper is shown to be a dimensional reduction of $N = 1$ three dimensional super-gauge theory in appendix A, where we also show that $N = 2$ super-gauge theory in two dimensions is a dimensional reduction of the $N = 1$ four dimensional super-gauge theory. Another set of MM equation and Ward identity is discussed in appendix B.

2. Equations of Motion of Large N Wilson Loops

As usual the Wilson line associated to a curve C_{xy} with end points at x and y is defined by

$$U(C_{xy}) = P \exp \left(i \int_x^y A_\mu dx^\mu \right),$$

which transforms under $A_\mu \to U A_\mu U^{-1} + i\partial_\mu U U^{-1}$ as $U(C_{xy}) \to U(x)U(C_{xy})U^{-1}(y)$. For a closed loop C_{xx}, $W(C_{xx}) = \frac{1}{N}\mathrm{tr}\ U(C_{xx})$ is gauge invariant. Its expectation value is what we want to calculate.

Another gauge invariant quantity relevant to our discussion is obtained by inserting the fermion field λ at two points x, y on the loop C. These two points divide the loop into two segments of curves C_{xy} and C'_{yx}. Now $W_\lambda(C_{xy}, C'_{yx})$ as a matrix is defined according to

$$\left(W_\lambda(C_{xy}, C'_{yx})\right)_{\alpha\beta} = \frac{1}{N}\mathrm{tr}\ \lambda_\alpha(x)U(C_{xy})\lambda_\beta(y)U(C'_{yx}),$$

W_λ is a two by two matrix.

One's first instinct is to write down the usual Makeenko-Migdal equation derived from the identity

$$\int [dA d\phi d\lambda] \mathrm{tr}\ \frac{\delta}{\delta A(x)} U(C_{xx}) e^{iS} = 0.$$

The equation is presented in appendix B. It is easy to see that this equation, unlike the MM equation in the pure gauge theory, will involve three different quantities. To get a closed set of equations, more Schwinger-Dyson equations are needed. Proceeding further,

one will soon realize that the number of equations will never terminate, namely a closed set of Schwinger-Dyson equations will involve infinitely many equations.

Supersymmetry plays an important role in this model. Clearly, if SUSY is not broken dynamically, there are many Ward identities one can write down. We shall prove in sect.4 that indeed SUSY is not broken, therefore for an arbitrary functional of fields $F(A,\phi,\lambda)$

$$\int[dAd\phi d\lambda]\delta_\epsilon(F(A,\phi,\lambda))e^{iS}=0, \tag{2.1}$$

where $\delta_\epsilon(F(A,\phi,\lambda))$ is the SUSY transformation of F. The reason behind the above identity is the following. First of all, the action is invariant under supersymmetry transformation, then the measure is invariant too[1]. If the vacuum is annihilated by the super-charges, then the boundary conditions in the path integral is also invariant under SUSY, thus

$$\int[dA^\epsilon d\phi^\epsilon d\lambda^\epsilon]F(A^\epsilon,\phi^\epsilon,\lambda^\epsilon)e^{iS}=\int[dAd\phi d\lambda]F(A,\phi,\lambda)e^{iS}$$

implying (2.1). If SUSY were broken, it would still be possible to write certain Ward identities. One would include effects of non-invariance of boundary conditions, such identities obtained may be called "anomalous" Ward identities.

Let $F=\mathrm{tr}\ \lambda(x)U(C_{xx})$ in (2.1), a Ward identity is readily written down, using transformation law (1.2):

$$\langle\mathrm{tr}\ F_{01}U(C_{xx})\rangle\sigma^1-\langle\mathrm{tr}\ D_\mu\phi(x)U(C_{xx})\rangle\sigma^\mu\sigma^3+2\oint dy^\mu\langle\mathrm{tr}\ \lambda(x)U(C_{xy})\lambda(y)U(C'_{yx})\rangle\bar\sigma_\mu=0, \tag{2.2}$$

where the last quantity is what we have already introduced. This is an equation of two by two matrix. Next, use the relation [11]

$$\frac{\partial}{\partial\sigma^{\mu\nu}(x)}\mathrm{tr}\ U(C_{xx})=i\mathrm{tr}\ F_{\mu\nu}U(C_{xx})$$

and the relation

$$\frac{1}{N}\mathrm{tr}\ D_\mu\phi(x)U(C_{xx})=\partial_\mu\frac{1}{N}\mathrm{tr}\ \phi(x)U(C_{xx})=\partial_\mu W_\phi(C_{xx}),$$

the Ward identity (2.2) is written as

$$-i\frac{\partial}{\partial\sigma(x)}W(C_{xx})\sigma^1-\partial_\mu W_\phi(C_{xx})\sigma^\mu\sigma^3+2\oint dy^\mu W_\lambda(C_{xy},C'_{yx})\bar\sigma_\mu=0, \tag{2.3}$$

[1] Without explicit calculation, the measure is invariant at least up the the first order in ϵ, since the Jacobian is bosonic and SUSY transformation is linear.

where we use $\partial/\partial\sigma(x)$ to denote $\partial/\partial\sigma^{01}(x)$, there is only one independent area element in two dimensions. The above is an equation relating three different quantities including the Wilson loop. To solve this equation, we expand the matrix W_λ as follows

$$W_\lambda(C_{xy}, C'_{yx}) = \sum_\mu W_\mu \sigma^\mu + W_2 i\sigma^2 + W_3\sigma^3. \tag{2.4}$$

Under Lorentz rotation $\lambda \to \exp(\theta\sigma^1)\lambda$, σ^μ transforms as a vector, σ^2 and σ^3 transform as a scalar. To keep Lorentz invariance, one then demands W_μ transform as a vector, W_2 and W_3 transform as a scalar. Under parity reflection $\lambda \to \sigma^3\lambda$, σ^3 is a scalar and σ^2 is a pseudo-scalar. Thus, W_3 is a scalar and W_2 is a pseudo-scalar. Substituting (2.4) into (2.3), one deduces

$$\frac{\partial}{\partial\sigma(x)} W(C_{xx}) = -2i \oint dy^\mu \epsilon_{\mu\nu} W^\nu(C_{xy}, C'_{yx}), \tag{2.5}$$

$$\partial_\mu W_\phi(C_{xx}) = 2\oint \left(\epsilon_{\mu\nu} dy^\nu W_2 + \eta_{\mu\nu} dy^\nu W_3\right), \tag{2.6}$$

$$\oint dy^\mu W_\mu = 0, \tag{2.7}$$

where the anti-symmetric tensor $\epsilon_{\mu\nu}$ is specified by $\epsilon_{01} = 1$. These equations are valid even for a finite N. The first two equations tell us that in order to calculate W and W_ϕ, it is enough to know W_λ. The last equation says that the total flux of W_μ along the loop is zero. This is important for us, it allows us to extend the quantity

$$\Phi(C_{xy}, C'_{yx}) = \int_x^y d\tilde{y}^\mu W_\mu(C_{x\tilde{y}}, C'_{\tilde{y}x}) \tag{2.8}$$

as a function of x and y into inside the loop C, by deforming the contour as shown below.

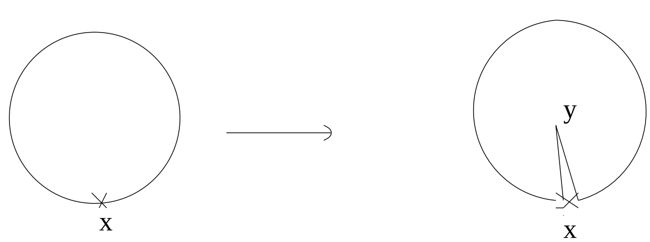

Figure 1

Note that this extension of Φ as a function of y depends on the location of x. The above argument is bit hand-waving. A more formal argument is the following. For a simple loop, a loop without intersection, Φ as a function of y when x fixed, is well-defined on the loop C. It is then always possible to analytically extend it onto the interior C. Again,

the extension depends on the position x. For a loop with intersection points, Φ may be multi-valued when y is at one intersection. The reason is that when it crosses around the loop, it will cross this intersection point at least twice. Formula (2.7) does not guarantee Φ be unique there. However, as we shall see, that Φ satisfies a differential equation which is not multi-valued anywhere when x itself is not an intersection point, and its solution can not be multi-valued anywhere. We thus believe that Φ is single valued even at an intersection point y, as long x is not that point.

From definition (2.8), it follows that $W_\mu = \partial_{y^\mu}\Phi$. Φ is well-defined globally, it follows that $\partial_0 W_1 - \partial_1 W_0 = 0$, consistent with eq.(2.7). Stokes formula when applied to (2.5) yields

$$\frac{\partial}{\partial\sigma(x)} W(C_{xx}) = 2i \int d\sigma(y)\partial_{y^\mu}\partial_{y_\mu}\Phi, \tag{2.9}$$

where the area integral extends to the domain enclosed by the loop C. Now it is desirable to derive an equation for Φ.

Instead of deriving the ordinary MM equation associated to translational invariance in the gauge field, we start with an equation associated to translational invariance in the fermionic field λ,

$$\int [dAd\phi d\lambda] \mathrm{tr}\ \frac{\delta}{\delta\lambda(x)} U(C_{xy})\lambda(y)U(C'_{yx})e^{iS} = 0. \tag{2.10}$$

The derivative when acts on $\lambda(y)$ gives rise to a delta function $\delta_{\alpha\beta}\delta^2(x-y)$ and a product tr $U(C_{xy})$tr $U(C'_{yx})$. Each factor in the product is not gauge invariant unless x coincides with y. This is guaranteed by the delta function factor. Two additional terms result from the action of the derivative on $\exp(iS)$:

$$\begin{aligned}\delta^2(x-y)\langle \mathrm{tr}\ U(C_{xy})\mathrm{tr}\ U(C'_{yx})\rangle + \frac{2N}{g^2}\bar\sigma^\mu\partial_\mu\langle \mathrm{tr}\ \lambda(x)U(C_{xy})\lambda(y)U(C'_{yx})\rangle \\ -\frac{2iN}{g^2}\sigma^3\langle \mathrm{tr}\ [\phi,\lambda](x)\lambda(y)U(C'_{yx})\rangle = 0.\end{aligned}$$

This equation of two by two matrix is valid for an arbitrary N. In the large N limit, apply the factorization theorem to the first term

$$\delta^2(x-y)W(C_{xy})W(C'_{yx}) + \frac{2}{g^2}\bar\sigma^\mu\partial_\mu W_\lambda - \frac{2i}{g^2}\sigma^3 W_{\phi\lambda} = 0, \tag{2.11}$$

where the new quantity

$$W_{\phi\lambda} = \frac{1}{N}\langle \mathrm{tr}\ [\phi,\lambda](x)U(C_{xy})\lambda(y)U(C'_{yx})\rangle.$$

To make use of the matrix equation (2.11), we make the expansion, as in (2.4)

$$W_{\phi\lambda} = \sum_{\mu} \tilde{W}_\mu \sigma^\mu + \tilde{W}_2 i\sigma^2 + \tilde{W}_3 \sigma^3. \tag{2.12}$$

Substituting this expansion and that in (2.4) into the matrix equation (2.11) and reading off coefficients of each basis matrix, we obtain

$$\partial^\mu W_\mu = -i\tilde{W}_3 + \frac{g^2}{2}\delta^2(x-y)W(C_{xy})W(C'_{yx}), \tag{2.13}$$

$$\epsilon^{\mu\nu}\partial_\mu W_\nu = i\tilde{W}_2, \tag{2.14}$$

$$\partial_1 W_3 - \partial_0 W_2 = i\tilde{W}_1, \tag{2.15}$$

$$\partial_1 W_2 - \partial_0 W_3 = -i\tilde{W}_0. \tag{2.16}$$

As we showed earlier, the vector W_μ is curl-less, so we conclude from (2.14) that $\tilde{W}_2 = 0$. This is a pseudo-scalar. Eqs.(2.15) and (2.16) do not respect Lorentz invariance unless $\partial_\mu W_2 = 0$. This implies that $W_2 = \mathrm{const}$. W_2 is also a pseudo-scalar, so if it is independent of positions x and y, the only reasonable constant is zero. Thus, $W_2 = \tilde{W}_2 = 0$. As far as W_μ is concerned, there is still an unknown quantity $\tilde{W}_3$ in (2.13). If one can show that this quantity is also vanishing, then eq.(2.13) together with the Ward identity (2.5) forms a closed system of equations for W_μ and W.

3. Solution of Wilson Loops

To make eq.(2.5) or (2.9) together with (2.13) a closed set of equations, the central problem is to determine $\tilde{W}_3$, a scalar quantity. It was already pointed out in the previous section that the pseudo-scalar $\tilde{W}_2 = 0$.

It is seen from the expansion (2.12) that a non-vanishing $\tilde{W}_2$ would have measured the disparity between the two off-diagonal elements of $W_{\phi\lambda}$. Its vanishing says that there is no disparity. Similarly, a non-vanishing $\tilde{W}_3$ measures the disparity between the two diagonal elements of $W_{\phi\lambda}$. It is natural to guess $\tilde{W}_3 = 0$. An exchange between the diagonal elements can be achieved by transformation

$$\lambda \to \sigma^1 \lambda, \tag{3.1}$$

which exchanges λ_1 and λ_2. Indeed this is a discrete symmetry of our theory (1.3), provided that a simultaneous transformation $\phi \to -\phi$ is made. This is because σ^3 changes sign under (3.1).

Now we can draw strong results from this symmetry. First consider W_λ, it should be invariant under the discrete symmetry (3.1). However, both σ^2 and σ^3 change sign under this transformation. It follows from expansion (2.4) that $W_2 = W_3 = 0$. We already argued for $W_2 = 0$ in the previous section with the help of eqs.(2.15) and (2.16). Substitute $W_2 = W_3 = 0$ into those equations, we find $\tilde{W}_\mu = 0$. This result can be obtained by observing the quantity $W_{\phi\lambda}$ too. Under the discrete transformation, $W_{\phi\lambda}$ changes its sign, however $\tilde{W}_\mu$ do not change their sign, therefore they must be zero. Unfortunately, we can not conclude $\tilde{W}_2 = \tilde{W}_3 = 0$ from this symmetry, for σ^2 and σ^3 do change their sign. Nevertheless, as a consequence of SUSY Ward identity and eq.(2.14), $\tilde{W}_2 = 0$. Thus, we have inferred that all coefficients in expansion of $W_{\phi\lambda}$ are zero except $\tilde{W}_3$.

We make the conjecture that $\tilde{W}_3 = 0$, which is very natural in our opinion. The fact that $\tilde{W}_2 = 0$ does not follow from any symmetry of the model encourages us to make this conjecture. It is plausible that $\tilde{W}_3 = 0$ is related to $\tilde{W}_2 = 0$ by certain duality. The latter is a pseudo-scalar while the former is a scalar, duality usually relates a scalar quantity to a pseudo-scalar quantity. For example, the electro-magnetic duality relates the vector E_i to the pseudo-vector B_i. We shall discuss another set of MM equation and Ward identity in appendix B, where we present an argument which is close to a proof of $\tilde{W}_3 = 0$.

It is possible that both $\tilde{W}_2$ and $\tilde{W}_3$ become singular when points x and y all approach an intersection point. But such complication will not alter our result obtained below, as long as we stay away from intersection points.

With $\tilde{W}_3 = 0$ and $W_\mu = \partial_{y^\mu}\Phi$, eq.(2.13) together with eq.(2.9) forms a simple system of equations

$$\begin{aligned} \frac{\partial}{\partial\sigma(x)}W(C_{xx}) &= 2i\int d\sigma(y)\partial_{y^\mu}\partial_{y_\mu}\Phi, \\ \partial_{y^\mu}\partial_{y_\mu}\Phi &= -\frac{g^2}{2}\delta^2(x-y)W(C_{xy})W(C'_{yx}). \end{aligned} \tag{3.2}$$

Now it is a simple matter to solve the Wilson loop from the above equations. One simply substitutes the second equation into the first one, and performs the area integral. If the loop C is smooth at x and not an intersection point, half of contribution of the delta function is picked up, because the area integral is restricted inside the loop, one then has

$$\frac{\partial}{\partial\sigma(x)}W(C_{xx}) = -\frac{ig^2}{2}W(C_{xx}). \tag{3.3}$$

The loop C may have many intersection points, therefore many windows. It is reasonable to assume that $W(C)$ depends on the loop only through areas of these windows. This is indeed dictated by (3.3). If the point x is on a segment of the loop separating a window

S_i from the infinite area outside the loop, a variation of $\delta\sigma(x)$ is simply a variation of area S_i. Eq.(3.3) results in

$$\partial_{S_i} W(S) = -\frac{ig^2}{2} W(S). \tag{3.4}$$

If the point x is sitting on a segment separating two windows S_i and S_j, and S_j is inside S_i, then

$$(\partial_j - \partial_i) W(S) = -\frac{ig^2}{2} W(S). \tag{3.5}$$

From the second equation in (3.2), $\Phi(x)$ is solved as $\Phi \sim \ln(x-y)^2 W(C_{xx})$, as long as x not an intersection point. Φ is singular at $y = x$. This singularity can be regularized as usual by introducing the $i\epsilon$ prescription. By definition, $W_\mu = \partial_{y^\mu}\Phi \sim (y-x)_\mu/(y-x)^2 W(C_{xx})$. This result can be contrasted with a perturbative consideration. Without coupling to the gauge field and the scalar, the first factor agrees with the usual Dirac propagator. The factorization when it is coupled to bosonic fields is interesting. We like to caution ourselves that this result may not be valid at an intersection point $x = y$. W_μ vanishes at a non-intersection point $y = x$, if the $i\epsilon$ prescription is used. This fact will be used in the discussion in appendix B.

Back to Wilson loops. Eqs.(3.4) and (3.5) are enough to determine the function $W(S)$. Let us consider a few examples. The simplest one is a simple loop without intersection point, as shown below. The solution to (3.4) is $W(S) \sim \exp(-\frac{ig^2}{2}S)$. The proportional coefficient must be one, for when S shrinks to a point $W = 1$. Note that the area law is exactly the same as in the pure gauge theory [2], although here we work in Minkowski spacetime.

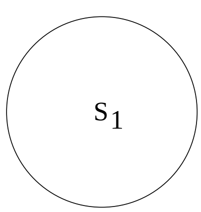

Figure 2

The next example is the 8 shaped curve. The two windows are not separated by a segment, we need use only (3.4). The result is

$$W(S_1, S_2) = \exp\left(-\frac{ig^2}{2}(S_1 + S_2)\right).$$

Again this is an area law agreeing with the pure gauge theory. So far both Wilson loops respect the Nambu string behavior.

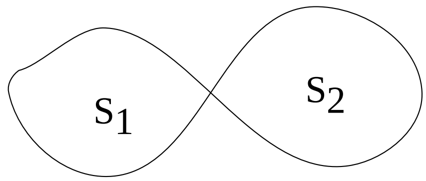

Figure 3

A crucial example is the loop in the next figure, where S_2 borders on S_1. Eq.(3.4) gives $\partial_1 W = -i(g^2/2)W$, this together with (3.5) gives $\partial_2 W = -ig^2 W$. Thus the solution to these equations is

$$W(S_1, S_2) = \exp\left(-\frac{ig^2}{2}(S_1 + 2S_2)\right) = \exp\left(-\frac{ig^2}{2}(S_1 + S_2 + S_2)\right).$$

There is no power dependence on S_2, unlike in the pure gauge theory. The above formula is perfectly in accordance with the Nambu string.

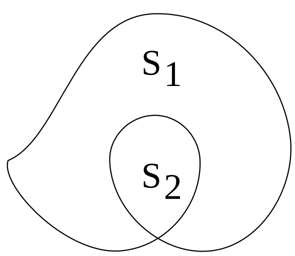

Figure 4

The final example is shown in the figure below. There are three windows. Our equations then determine the Wilson loop

$$\begin{aligned} W(S_1, S_2, S_3) &= \exp\left(-\frac{ig^2}{2}(S_1 + 2S_2 + 3S_3)\right) \\ &= \exp\left(-\frac{ig^2}{2}(S_1 + S_2 + S_3 + S_2 + S_3 + S_3)\right), \end{aligned}$$

also agrees with the Nambu string.

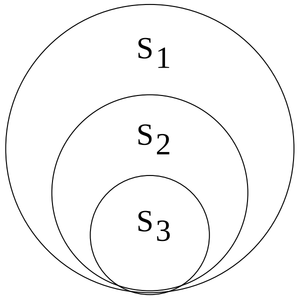

Figure 5

It is not hard to convince oneself that the standard area law should persists for all kinds of loops. It is therefore very interesting to learn that in a theory of more complicated spacetime physics, the world sheet picture is simpler. Certainly we are not claiming here that the whole theory is described by the Nambu string without folds, since there are many other physical operators independent of Wilson loops.

4. Hamiltonian Formalism and the Witten Index

We discuss the Hamiltonian formalism in this section for two purposes. First, we want to determine whether supersymmetry is dynamically broken. To adapt an argument of Witten in [10] to calculate to Witten index, we need to put the system into a spatial box of finite length. Second, to study nonperturbative effects, it is often tempting to put a system on a lattice. If one wants to make use of supersymmetry, this can not be normally done on a full spacetime lattice, since supersymmetry implies translation invariance in two directions. However, some models can be put on a spatial lattice without spoiling a subset of supersymmetry generators, provided no spatial translation is generated by these generators. This can be done in two dimensions for $N=1$ supersymmetry. In four dimensions, one requires at least $N=2$ [12].

For a gauge theory, there is one more complication. One need to fix a gauge in the Hamiltonian formalism. A gauge must be chosen such that it is invariant under some of supersymmetry transformations. In the two dimensional $N=1$ Yang-Mills theory at hand, there are two supersymmetry generators Q_α, each is Hermitian and satisfies $Q_\alpha^2=2H$, H is the Hamiltonian. One can not keep both Q_α, since the anti-commutator of the two generators gives rise to the spatial translation generator. Now if one chooses the temporal gauge $A_0=0$, it is easy to see from the transformation law (1.2) that no supersymmetry survives this gauge. However

$$\delta(A_0+\phi)=2i\lambda(\sigma^3-1)\epsilon=-4i\lambda_2\epsilon_2,$$

so if $\epsilon_2 = 0$, the combination $A_0 + \phi$ is invariant under Q_1. We thus fix a gauge in which $A_0 + \phi = 0$. With this gauge choice

$$S = \frac{N}{g^2} \int d^2x \mathrm{tr} \left(\frac{1}{2}(\partial_0 A)^2 + \partial_0 A D_x \phi + \frac{1}{2}(\partial_0 \phi)^2 + i\lambda\partial_0\lambda + i\lambda\sigma^1 D_x \lambda + \lambda(1-\sigma^3)[\phi, \lambda] \right), \tag{4.1}$$

where A is the spatial component of the gauge field, and D_x is the spatial covariant derivative. The Yukawa coupling involves only λ_2 in the above action.

The canonical momenta are read off from the action

$$\begin{aligned} \Pi_A &= \frac{N}{g^2} F_{01} = \frac{N}{g^2} (\partial_0 A + D_x \phi), \\ \Pi_\phi &= \frac{N}{g^2} \partial_0 \phi, \quad \Pi_\lambda = i\frac{N}{g^2}\lambda. \end{aligned} \tag{4.2}$$

The Hamiltonian is then

$$H = \frac{N}{g^2} \int dx \mathrm{tr} \left(\frac{1}{2}(\partial_0 A)^2 + \frac{1}{2}(\partial_0 \phi)^2 - i\lambda\sigma^1 D_x\lambda + \lambda(\sigma^3 - 1)[\phi, \lambda] \right). \tag{4.3}$$

It is also straightforward to write down the unbroken super-charge Q_1

$$Q_1 = 2 \int dx \mathrm{tr} \left(\lambda_1 \Pi_\phi + \lambda_2 (\Pi_A - \frac{N}{g^2} D_x \phi) \right). \tag{4.4}$$

To check the relation $Q_1^2 = 2H$, one should notice the fact that since Π_λ is the same as λ, the anti-commutator is the half the value of the usual anti-commutator. Specifically,

$$\{\lambda^a_\alpha(x), \lambda^b_\beta(y)\} = \frac{g^2}{2N} \delta_{\alpha\beta} \delta(x-y).$$

We will not try to write down the Hamiltonian in terms of the link variable and other fields, except making a comment on the role of fermions. There will be a doubling problem as usual on a spatial lattice. Here one solves the problem by putting λ_1 on even sites, and λ_2 on odd sites, much like what is done in [12]. In the super-charge (4.4), although Π_ϕ will be only assigned on even sites, the term $D_x\phi$ involves ϕ both at an even site and an odd site, it is easy to check that the relation $Q_1^2 = 2H$ is satisfied by this prescription of solving the doubling problem. The continuum limit is naturally achieved.

In the remaining part of this section, we calculate the Witten index. We follow closely a calculation by Witten of the index in the four dimensional super-gauge theory in [10]. We put the system into a spatial box with boundary $x = 0, L$, with periodic boundary conditions. If the Witten index is nonvanishing for all finite L, it is certainly nonvanishing in the infinite volume. One may consider a gauge group $U(N)$. But since all fields are

in adjoint representation, the $U(1)$ sector is free and then does not affect the issue of supersymmetry breaking. So we will consider gauge group $SU(N)$ (If there is a matter sector, the $U(1)$ sector can not be ignored, and indeed is a subtle problem as investigated in [10].) Unlike in four dimensions, it seems that the weak coupling does not come to our rescue, the reason is that g^2 has a dimension of mass squared, therefore the meaning of weak coupling is senseless. Still, we have a dimensionless combination g^2L^2. When L held fixed, we can make this combination arbitrarily small. If the Witten index tr $(-1)^F$ is not zero for a small g^2L^2, then it is not zero for arbitrary g^2L^2 since the index is invariant under arbitrary deformation of parameters.

Now the trick of Witten consists in reducing the problem to a problem of zero modes (with vanishing momentum). Any nonzero momentum mode will have a energy greater than $1/L$. For a zero mode, we will see that an excited state associated to the zero modes of A has a energy g^2L, which is much smaller than $1/L$ if g^2L^2 is small enough. Thus it is safe to ignore nonzero momentum modes in the discussion. Write the Hamiltonian in terms of canonical momenta of fields

$$\begin{aligned} H = \frac{g^2}{N} \int dx \mathrm{tr} \ \left(\frac{1}{2}\Pi_A^2 + \frac{1}{2}\Pi_\phi^2\right) - \int dx \mathrm{tr} \ \Pi_A D_x \phi \\ + \frac{N}{g^2} \int dx \mathrm{tr} \ \left(\frac{1}{2}(D_x\phi)^2 - i\lambda D_x \lambda + \lambda(\sigma^3 - 1)[\phi, \lambda]\right). \end{aligned} \tag{4.5}$$

The last term of (4.5) tells us that the zero modes satisfy

$$D_x\phi = 0, \quad D_x\lambda + i(\sigma^3 - 1)[\phi, \lambda] = 0. \tag{4.6}$$

We still have freedom to do spatial gauge transformation, and it is always possible to gauge transform A into a constant matrix $A(0)$. This constant matrix can not be gauged away in general, since the Wilson line tr $\exp(iA(0)L)$ can not be gauged away with periodic gauge transformation. Because only the zero mode A is left, the second term in (4.5) is independent of non-zero momentum modes of ϕ. This is why we can focus our attention on the zero mode of ϕ in the first place. Now the solution to the first equation in (4.6) is given by

$$\phi(x) = e^{-iA(0)x}\phi(0)e^{iA(0)x}.$$

The periodic boundary condition $\phi(L) = \phi(0)$ implies that $[\phi(0), \exp(iA(0)L)] = 0$. Thus, the Hermitian matrix and the unitary matrix $exp(iA(0)L)$ can be simultaneously diagonalized, which in turn implies $[A(0), \phi(0)] = 0$, therefore $\phi(x) = \phi(0)$. With remaining constant gauge transformations, we can always put $A(0)$ and $\phi(0)$ into a maximal Abelian

sub-algebra of $SU(N)$. Let $A(0) = \sum_a A_a t^a$, $\phi(0) = \sum_a \phi_a t^a$. t^a are generators of this sub-algebra.

The second equation of (4.6) can be solved just as the first one, and the result is that $\lambda_\alpha(0) = \sum_a \lambda^a_\alpha t^a$. Now apparently there is a vacuum which is annihilated by Π_A and Π_ϕ. Let it denoted by $|\Omega\rangle$. The fermion part form a Clifford algebra $\{\lambda^a_\alpha, \lambda^b_\beta\} = 1/2\delta_{\alpha\beta}\delta_{ab}$. Its representation is 2^{N-1} dimensional. To construct this representation, let $\lambda^a = \lambda^a_1 - i\lambda^a_2$, and $\bar\lambda^a = \lambda^a_1 + i\lambda^a_2$. Now $\{\lambda^a, \lambda^b\} = \{\bar\lambda^a, \bar\lambda^b\} = 0$ and $\{\lambda^a, \bar\lambda^b\} = \delta_{ab}$. We let $|\Omega\rangle$ also be the vacuum annihilated by λ^a, then any other state can be written as $\bar\lambda^{a_1} \dots \bar\lambda^{a_i}|\Omega\rangle$.

To count the true vacua, our final ingredient is the residual gauge group. First, one can do gauge transformation $U = \exp(i2\pi t^a x/L)$. It is periodic and shifts A_a by an amount of $2\pi/L$. So A_a is a periodic variable. While ϕ_a is not restricted. Second, there is global gauge group consisting of global gauge transformations mapping the maximal Abelian sub-algebra into itself and is called the Weyl group. For $SU(N)$, it is the permutation group S_N. Any physical vacuum is invariant under a permutation. Since $\Pi_{A_a} = -i1/L\partial_{A_a}$ and A_a is periodic with a period $2\pi/L$, an excited state, according to (4.5), has an energy g^2L. As long as g^2L is much smaller than $1/L$, non-zero momentum modes can be safely ignored.

The problem is that there are excited states of arbitrarily small energy, for ϕ_a's are not restricted. One may put a cut-off on the space of ϕ_a, say tr $\phi^2(0) = \sum_a \phi_a^2 \le \Lambda^2$. This cut-off is gauge invariant and will cause no trouble to have the theory well-defined. Now let Λ to be large enough such that g^2/Λ^2 is much smaller than 1 (so that the excited state associated to ϕ_a has an energy $g^2/(\Lambda^2 L)$ much smaller than $1/L$), then the spectrum is discrete, the Witten index is well-defined. In the end of calculation, one can push Λ to infinity without changing the Witten index.

One can either assume that $|\Omega\rangle$ is invariant under permutations, or pseudo-invariant (changes its sign under a odd permutation). If it is invariant, then no more invariant states can be constructed. This can be shown along the line of [10]. In this case tr $(-1)^F = 1$. If one assumes $|\Omega\rangle$ be pseudo-invariant, an invariant state can be generated by acting on it by the pseudo-invariant operator $\bar\lambda^1 \dots \bar\lambda^{N-1}$. This state , call it $|\tilde\Omega\rangle$, has the statistics $(-1)^N$, if one assumes that the pseudo-invariant state $|\Omega\rangle$ is fermionic. The Witten index is then tr $(-1)^F = (-1)^N$. This is an ambiguity hard to resolve in the four dimensional super-gauge theory [10]. We argue that this issue can be resolved in our model. Note that if $|\Omega\rangle$ is invariant, then $|\tilde\Omega\rangle$ is pseudo-invariant and has a statistics $(-1)^{N-1}$. This state is annihilated by all $\bar\lambda$, so as a state it should be treated on the equal footing as $|\Omega\rangle$. This means that when $|\Omega\rangle$ is pseudo-invariant and $|\tilde\Omega\rangle$ is invariant, the latter should be regarded as a bosonic state, and the former has a statistics $(-1)^{N-1}$ not -1 as we previously assumed. So the Witten index is 1 in this case too.

We thus have shown that the Witten index is nonvanishing for a finite L and small g^2L^2, therefore it is nonvanishing for arbitrary L and g^2. Supersymmetry is not broken in the 2D super-gauge theory. If one further demands the existence of vacuum in the limit $N=\infty$ and its statistics being well-defined, one has to choose tr $(-1)^F=1$. The other choice $(-1)^N$ does not make any sense.

5. Discussion

Tracing the reason why the equations of motion in the super-gauge theory are considerably simpler than those in the bosonic theory, we realize that the Schwinger-Dyson equation associated to translation of the fermion field is a first order differential equation, since the kinetic term of the fermion in the action is of the first order. The Ward identity (2.3) relates the Wilson loop to the loop with two insertions of the fermion field in a simple manner. Recall that much difficulty in dealing with the ordinary MM equation arises from the second order derivative of the Wilson loop in the equation, which is absent in equations discussed in sect.2.

We have seen that the solution of the Wilson loop is really the Nambu string without folds. One would like to proceed further to study other physical observables, in order to learn more about the string theory underlying this super-gauge theory. Evidently, the string theory possesses a spacetime supersymmetry, and the physical spectrum should furnish a representation of SUSY.

The power of combining the Schwinger-Dyson equations, in the guise of loop equations, and Ward identities associated to SUSY is manifest in our model. We have studied one set of these equations in the previous sections. In appendix B, we shall study another set of equations, where interesting results are also obtained. Although the method we present in this paper is markedly different from the holomorphy technique and duality argument of Seiberg-Witten, we suspect that there is intimate relationship. This may become evident if we study a high dimensional super-gauge theory, for only there duality also enters into loop variables [13]. The $N=1$ 2D super-gauge theory is a dimensional reduction of a $N=1$ 3D super-gauge theory (appendix A), thus with additional input, hopefully the large N problem in this model can also be solved. We plan to study this model in the future.

A more straightforward application of considerations here would be to the $N=2$ 2D super-gauge theory. As shown in appendix A, there is a complex scalar in this model, and one more Majorana spinor in the adjoint representation. Classically, there are many vacua, characterized by a moduli space, very similar to $N=2$ four dimensional theories studied

in [5]. And the kind of "duality" suggested in sect.3 becomes obvious in this model. Also, the $N = 1$ 4D super-gauge theory dimensionally reduces to this model, one may learn things in the high dimensional model by studying this 2D model.

Finally, we have studied only the Wilson loop and a couple of related physical observables. Physical problems such as the spectrum in this model are still open. To solve them, one would need study more physical observables, a systematic scheme would be valuable. Indeed a possible such scheme, the free variable representation of master fields, has been the subject of a flurry of recent activities [14] - [18]. The master field was constructed by Singer for the 2D pure gauge theory. The relative ease in constructing it is due to the freeness of the master field in a special gauge. The model studied in this paper then presents a challenge: The master field is no longer expected to be free for different momentum modes.

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Appendix A

The $N = 1$ four dimensional super-gauge theory without matter contains a vector super-multiplet, in which there are a gauge field A_m and its super-partner λ. Here following conventions in [9] Latin letters m and n are used to denote the spacetime index. In the so-called Wess-Zumino gauge, there is an auxiliary field D, which is also in the adjoint representation. The action

$$S = \int d^4x \mathrm{tr} \left(-\frac{1}{4} F_{mn}^2 - i\bar{\lambda}\bar{\sigma}^m D_m \lambda + \frac{1}{2} D^2 \right) \tag{A.1}$$

is invariant under the SUSY transformation

$$\begin{aligned} \delta A_m &= -i\bar{\lambda}\bar{\sigma}_m \epsilon + i\bar{\epsilon}\bar{\sigma}_m \lambda, \\ \delta\lambda &= \sigma^{mn} F_{mn} \epsilon + iD\epsilon, \\ \delta D &= D_m \bar{\lambda}\bar{\sigma}^m \epsilon + \bar{\epsilon}\bar{\sigma}^m D_m \lambda. \end{aligned} \tag{A.2}$$

A three dimensional action is readily obtained by dropping out x^2 as well as A_2. To have a supersymmetric theory, one demands that λ is a Hermitian matrix, instead a complex one. It is easy to see that this is consistent with transformation (A.2), where ϵ also becomes a real spinor and $\delta A_2 = 0$. In addition, $\delta D = 0$, since all σ^m except σ^2 are symmetric. Thus, the auxiliary field can be dropped out. Now, our two dimensional super-gauge theory as given in (1.1) is simply a dimensional reduction of the three dimensional super-gauge theory.

In going from four dimensions to three dimensions, we dropped out A_2 and half of degrees of freedom in λ. If one does not do so, a super-gauge theory can still be obtained, where A_2 becomes a scalar in three dimensions, and λ decomposes into two Majorana fermions, $\lambda = \lambda_1 + i\lambda_2$. This is a $N = 2$ super-gauge theory in three dimensions, and the auxiliary field is also kept ($\delta D \neq 0$). Reducing one more dimension x^3, a $N = 2$ super-gauge theory in two dimensions is obtained. The field content is: A gauge field, two Majorana spinors, one scalar $\phi_1 = A_3$ and one pseudo-scalar $\phi_2 = A_2$. Since the action is a direct reduction of (A.1) into two dimensions, we will not write it down here. This is a model of great interest to study, along the line of this paper. The "duality" we mentioned in sect.3 becomes apparent in this theory, the exchange of role of ϕ_1 and ϕ_2. If one forms a complex scalar from these two field, then the duality transformation is just a complex conjugate transformation. Two spinor fields will also get exchanged too. Classically, there are many vacua corresponding to different expectation values of the complex scalar. So this model is similar to the $N = 2$ super-gauge theories in four dimensions. In addition, as we have seen, this is a dimensionally reduced model of $N = 1$ 4D super-gauge theory. So if one wishes to probe some physics in this 4D theory, the $N = 2$ 2D super-gauge theory should serve as a good starting point.

Appendix B

A Schwinger-Dyson equation associated to translation of the fermion field is considered in the main body of this paper. It is not the original Makeenko-Migdal equation, which is associated to translation of the gauge field. It is easy to generalize the ordinary MM equation in the pure gauge theory to our model, which derives from

$$\int [dAd\phi d\lambda] \mathrm{tr}\ \frac{\delta}{\delta A(x)} U(C_{xx}) e^{iS} = 0.$$

Taking terms of the fermion field and of the scalar field in the action into account

$$\begin{aligned}&\langle \mathrm{tr}\ D_\nu F^{\nu\mu}(x) U(C_{xx})\rangle - i\langle \mathrm{tr}\ [\phi, D^\mu \phi] U(C_{xx})\rangle - 2\langle \mathrm{tr}\ \lambda \bar{\sigma}^\mu \lambda(x) U(C_{xx})\rangle \\ &+ g^2 N \int dy^\mu \delta^2(x-y) \langle \mathrm{tr}\ U(C_{xy}) \mathrm{tr}\ U(C'_{yx})\rangle = 0,\end{aligned} \qquad (B.2)$$

where the integral in the last term is taken as properly regularized when $x = y$: The contribution of the delta function is ignored except when $x = y$ is an intersection point and both $U(C_{xy})$ and $U(C'_{yx})$ are nontrivial. Applying the factorization theorem in the large N limit to the last term in (B.2)

$$\partial_\nu \frac{\partial}{\partial \sigma^{\nu\mu}(x)} W(C_{xx}) + W^\mu_\phi(C_{xx}) + 2iW^\mu(C_{xx}) + ig^2 \oint dy^\mu \delta^2(x-y) W(C_{xy}) W(C'_{yx}) = 0, \qquad (B.2)$$

where

$$W^\mu_\phi = \frac{1}{N} \langle \mathrm{tr}\ [\phi, D^\mu \phi](x) U(C_{xx})\rangle$$

and $W^\mu(C_{xx})$ is a term of W_λ in the expansion (2.4) and when $C'_{yx} = 0$. In addition to the Wilson loop, the generalized MM equation involves two additional quantities. When x is not an intersection point, the last term does not contribute, for the integral is regularized [2]. Now $W(C_{xx})$ depends only on areas of windows of the loop C, the derivative $\partial/\partial\sigma(x) W(C_{xx})$ must be independent of x in the vicinity of x, as long as x is not an intersection point [2]. We deduce from the MM equation (B.2) the following identity

$$W^\mu_\phi(C_{xx}) + 2iW^\mu(C_{xx}) = 0 \qquad (B.3)$$

This equation will play a crucial role in checking the consistency of our conjecture $\tilde{W}_3 = 0$ below. We expect that (B.3) is no longer true when x is an intersection point, otherwise

[2] If x is an intersection point, this derivative is discontinuous across x, since the area derivative on a different side involves different window areas.

the MM equation (B.2) would be identical to the one in the pure gauge theory, where the solution for $W(C)$ is completely different from what we obtained in sect.3.

Next consider another Ward identity associated to supersymmetry. Substitute $F = \mathrm{tr}\,[\phi, \lambda](x)U(C_{xx})$ into the general identity (2.1) and make use of the SUSY transformation (1.2), we obtain a two by two matrix equation

$$\begin{aligned}&\langle \mathrm{tr}\,[\phi, F_{01}](x)U(C_{xx})\rangle\sigma^1 - \langle \mathrm{tr}\,[\phi, D_\mu\phi](x)U(C_{xx})\rangle\sigma^\mu\sigma^3 - 4i\langle \mathrm{tr}\,\lambda\lambda(x)U(C_{xx})\rangle\sigma^3\\ &+2\oint\langle \mathrm{tr}\,[\phi, \lambda](x)U(C_{xy})\lambda(y)U(C'_{yx})\rangle\bar{\sigma}_\mu = 0.\end{aligned} \tag{B.4}$$

All the terms except the first term are familiar quantities. The first term is vanishing, if x is not an intersection point. The reason is simple. Note that $\mathrm{tr}\,[\phi, F_{01}](x)U(C_{xx}) = \mathrm{tr}\,(\phi F_{01}(x)U(C_{xx}) - \phi(x)U(C_{xx})F_{01}(x))$, and the effect of $F_{01}(x)$ when acting on $U(C_{xx})$ is an area derivative, whether it acts from the left or right, so this quantity is identically zero. After this term is dropped out from the above equation, the equation is rewritten in terms of quantities defined before

$$2\oint dy^\mu W_{\phi\lambda}(C_{xy}, C'_{yx})\bar{\sigma}_\mu = 4iW_\lambda(C_{xx})\sigma^3 + W^\mu_\phi(C_{xx})\sigma_\mu\sigma^3. \tag{B.5}$$

Remind ourselves that this equation is valid only when x is not an intersection point. Now we are in a position to check the conjecture $\tilde{W}_3 = 0$. As argued in sect.3, the nonvanishing terms in the expansion of W_λ in terms of sigma matrices are W_μ. They are vanishing too, if the segment $C'_{yx} = 0$ and properly regularized, see sect.3. From this fact and (B.3), it follows that $W^\mu_\phi(C_{xx}) = 0$. We then see that the r.h.s. of (B.5) is zero, which in turn implies

$$\oint dy^\mu \left(\tilde{W}_2 i\sigma^2 + \tilde{W}_3\sigma^3\right)\bar{\sigma}_\mu = 0,$$

where we used the expansion (2.12) and the result $\tilde{W}_\mu = 0$. This equation is consistent with $\tilde{W}_2 = \tilde{W}_3 = 0$. Indeed this is not even too much a weaker equation, as it might appear. It is equivalent to the following two equations

$$\begin{aligned}\oint dy^0\tilde{W}_2 - \oint dy^1\tilde{W}_3 = 0,\\ \oint dy^0\tilde{W}_3 - \oint dy^1\tilde{W}_2 = 0.\end{aligned}$$

Remember that $\tilde{W}_3$ is a scalar and $\tilde{W}_2$ is a pseudo-scalar, the above equations strongly suggest that these quantities are constant. If one further inserts the known result $\tilde{W}_2 = 0$ into above equations, then $\oint dy^0\tilde{W}_3 = \oint dy^1\tilde{W}_3 = 0$.

The conclusion from the study of another set of MM equation and Ward identity is that $\tilde{W}_3 = 0$ is altogether a reasonable assumption. Indeed we believe that our argument presented above is close to a proof. The fact that both (B.3) and (B.5), which have been crucial for our consistency check, are valid only when x is not an intersection point, cautions us that $\tilde{W}_3 = 0$ may be not true at an intersection point. Luckily, for our solution of the Wilson loop in sect.3, we do not need touch intersection points.

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